Two Proofs of Cauchy's Theorem.

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The following proofs of Cauchy's theorem that $\int_1^2 wdz$ has the same value for any two paths joining the points 1 and 2, provided that w is a uniform and continuous function of $z\,(= x + i)$ throughout the area included between those paths, are very simple; and I think they have the merit of more pointedly turning on the fundamental property that dw/dz is independent of the direction of dz than do the proofs usually given.

1°. Let the integral be taken along a certain path, and let the path be slightly deformed; denote the effect of this deformation by δ; then

$$\begin{aligned}\delta\int wdz &= \int\delta\,(wdz) = \int\delta w\,.\,dz + \int w\,.\,\delta\,(dz) = \int\delta w\,.\,dz + \int wd\,(\delta z)\\ &= [w\delta z]_1^2 + \int(\delta w\,.\,dz - \delta z\,.\,dw)\\ &= \int(\delta w\,.\,dz - \delta z\,.\,dw),\end{aligned}$$

since δz is zero at the points 1 and 2. All this is true whether w be a function of z or merely a function of x and y. But if w is a function of z, uniform and continuous in the region in question, $\delta w/\delta z = dw/dz$, or $\delta w\,.\,dz - \delta z\,.\,dw = 0$; hence the deformation does not affect the integral. This proves the theorem.

2°. Consider an infinitesimal contour, c, containing the point z_0. At the point z_0 let $w = w_0$, and put $z = z_0 + \zeta$, $w = w_0 + \omega$. Then

$$\begin{aligned}\int_c wdz &= \int_c (w_0 + \omega)\,d\zeta = w_0\int_c d\zeta + \int_c \omega d\zeta\\ &= \int_c \omega d\zeta,\end{aligned}$$

since $\int_c d\zeta$ is obviously 0. Now, if w is a function of z (uniform and continuous

in the region considered), ω/ζ is constant, $=A$ say, around the contour by the fundamental property already referred to; hence

$$\int_c wdz = A\int_c \zeta d\zeta = \frac{1}{2}A\int_c d(\zeta^2) = 0.$$

Thus the integral taken around an infinitesimal contour vanishes. Hence, by addition, the integral taken around any contour vanishes, and the theorem is proved.

Strictly speaking, what was proved about $\int_c wdz$ is not that it is absolutely 0, but that it is an infinitesimal of a higher order than the second; but this is of course sufficient for the purpose.

It may be added that neither of these proofs depends on the fact that $z=x+iy$; they are equally applicable if z is any function of x and y, and w a function of z; z and w being supposed uniform and continuous in the region concerned.

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